Indian Statistical Institute, Bangalore

B. Math. First Year First Semester

Analysis -I

Final Examination — Date: 13-11-2017

Maximum marks: 100 — Time: 3 hours

Instructor: B V Rajarama Bhat

(1) Let $\{b_n\}_{n\geq 1}$ be a sequence defined by $b_1 = -20$ and

$$b_{n+1} = 1 - \sqrt{1 - b_n} \text{ for } n \geq 1.$$

Show that the sequence $\{b_n\}_{n\geq 1}$ is convergent. Find its limit. [15]

(2) Let $a < b$ and $c < d$ be real numbers. Suppose $f : [a, b] \to [c, d]$ is a continuous bijection. Show that f^{-1} is continuous. [15]

(3) Let $\{x_n\}_{n\geq 1}$ be a bounded sequence of real numbers and let

$$M = \limsup_{n\to\infty} x_n.$$

Show that there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that

$$\lim_{k\to\infty} x_{n_k} = M.$$

[15]

(4) Let $h : [-10, 10] \to \mathbb{R}$ be a differentiable function satisfying $h(-10) = -10$ and $h(10) = 10$. Suppose $h'(x) \leq 1$ for all $x \in [-10, 10]$. Show that $h(x) = x$ for all x. (Hint: Consider the function $g(x) = x - h(x)$.) [15]

(5) State and prove Rolle's theorem. [15]

(6) Let $a < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be a function. Then f is said to satisfy a **Lipschitz condition** if there is a positive real number K such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in [a, b]$.

(i) Give an example of a function not satisfying a Lipschitz condition.

(ii) Suppose $f : [a, b] \to \mathbb{R}$ is differentiable and the derivative f' is continuous on $[a, b]$. Show that f satisfies a Lipschitz condition. [15]

(7) Show that if a series of real numbers $\sum_{n=1}^{\infty} a_n$ converges absolutely then $\sum_{n=1}^{\infty} a_n^2$ converges absolutely. Show that the converse is not true in general. [15]